R. P. BURN

NUMBERS AND FUNCTIONS

STEPS INTO ANALYSIS

Second edition

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

© Cambridge University Press 1992, 2000

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 1992
Second edition 2000
Reprinted 2001

Printed in the United Kingdom by Athenaeum Press Ltd, Gateshead, Tyne and Wear

Typeset in Times 10/12.5pt [VN]

A catalogue record for this book is available from the British Library

ISBN 0 521 78836 6 paperback

Contents

Preface to first edition xii
Preface to second edition xxi
Glossary xxii

PART I NUMBERS

1 Mathematical induction 3
Mathematical induction (qns 1–8) 3

Historical note 6
Answers and comments 8

2 Inequalities 10
Positive numbers and their properties (qns 1–29) 10
Summary – properties of order 14

Arithmetic mean and geometric mean (qns 30–39) 14
Completing the square (qns 40–42) 16
The sequence $(1 + 1/n)^n$ (qns 43–49) 17
nth roots (qns 50, 51) 18
Summary – results on inequalities 18

Absolute value (qns 52–64) 19
Summary – results on absolute value 21

Historical note 21
Answers and comments 22

3 Sequences: a first bite at infinity 28
Monotonic sequences (qn 4) 29
Bounded sequences (qns 5–7) 30
Subsequences (qns 8–16) 31

Sequences tending to infinity (qns 17, 18) 35
Archimedean order and the integer function (qns 19–23) 36
Summary – the language of sequences 37

Null sequences (qns 24–47) 38
Summary – null sequences 44

Convergent sequences and their limits (qns 48–60) 45
Boundedness of convergent sequences (qns 61–63) 49
Quotients of convergent sequences (qns 64–69) 49
d'Alembert's ratio test (qns 70–74) 50
Convergent sequences in closed intervals (qns 75–80) 52
Intuition and convergence (qns 81–83) 53
Summary – convergent sequences 55

Historical note 57
Answers and comments 59

4 Completeness: what the rational numbers lack 68
The Fundamental Theorem of Arithmetic (qns 1–3) 68
Dense sets of rational numbers on the number line (qns 4–10) 69
Infinite decimals (qns 11–17) 70
Irrational numbers (qns 18–21) 72
Infinity: countability (qns 22–30) 73
Summary 75

The completeness principle: infinite decimals are convergent (qns 31, 32) 76
Bounded monotonic sequences (qns 33–36) 78
*n*th roots of positive numbers, *n* a positive integar (qns 37–41) 79
Nested closed intervals (qn 42) 80
Convergent subsequences of bonded sequences (qns 43–47) 81
Cluster points (the Bolzano–Weierstrass theorem) (qns 48–54) 81
Cauchy sequences (qns 55–58) 83
Least upper bounds (sup) and greatest lower bounds (inf) (qns 59–82) 85
Upper bounds and greatest terms (qns 59–61) 85
Least upper bound (sup) (qns 62–66) 86
Lower bounds and least members (qns 67–70) 87
Greatest lower bound (inf) (qns 71–78) 88
sup, inf and completeness (qns 79–82) 89
lim sup and lim inf (qns 83, 84) 90
Summary – completeness 91

Historical note 92

Answers and comments 95

5 Series: infinite sums 104
Sequences of partial sums (qns 1–10) 104
The null sequence test (qns 11–13) 107
Simple consequences of convergence (qns 14–22) 107
Summary – convergence of series 108

Series of positive terms 109
First comparison test (qns 23–29) 109
The harmonic series (qn 30) 110
The convergence of $\sum 1/n^{\alpha}$ (qns 31, 32) 110
Cauchy's *n*th root test (qns 33–39) 111
d'Alembert's ratio test (qns 40–50) 112
Second comparison test (qns 51–55) 113
Integral test (qns 56–61) 114
Summary – series of positive terms 116

Series with positive and negative terms 117
Alternating series test (qns 62–65) 117
Absolute convergence (qns 66–70) 118
Conditional convergence (qn 71) 119
Rearrangements (qns 72–77) 120
Summary – series of positive and negative terms 122

Power series 123
Application of d'Alembert's ratio test and Cauchy's *n*th root test for absolute convergence (qns 78–90) 123
Radius of convergence (qns 91–101) 123
Cauchy–Hadamard formula (qns 102–107) 125
The Cauchy product (qns 108–113) 126
Summary – power series and the Cauchy product 128

Historical note 129
Answers and comments 131

PART II FUNCTIONS

6 Functions and continuity: neighbourhoods, limits of functions 143
Functions (qn 1) 143
The domain of a function 143
The range and co-domain of a function (qns 2–4) 144
Bijections and inverse functions (qns 5–6) 145
Summary – functions 145

Continuity (qns 7–11) 146
Definition of continuity by sequences (qns 12–18) 147
Examples of discontinuity (qns 19, 20) 148
Sums and products of continuous functions (qns 21–31) 149
Continuity in less familiar settings (qns 32–35) 150
A squeeze rule (qns 36, 37) 151
Continuity of composite functions and quotients of continuous functions (qns 38–55) 151
Summary – continuity by sequences 154

Neighbourhoods (qns 56–63) 154
Definition of continuity by neighbourhoods (qns 64–72) 156
One-sided limits 159
Definition of one-sided limits by sequences (qns 73–82) 159
Definition of one-sided limits by neighbourhoods (qns 83–85) 162
Two-sided limits 163
Definition of continuity by limits (qns 86–92) 163
Theorems on limits (qns 93–99) 164
Limits as $x \to \infty$ and when $f(x) \to \infty$ (qns 100, 101) 166
Summary – continuity by neighbourhoods and limits 167

Historical note 168
Answers 171

7 Continuity and completeness: functions on intervals 182
Monotonic functions: one-sided limits (qns 1–7) 182
Intervals (qns 8–11) 183
Intermediate Value Theorem (qns 12–21) 185
Inverses of continuous functions (qns 22–28) 186
Continuous functions on a closed interval (qns 29–36) 188
Uniform continuity (qn 37–45) 190
Extension of functions on $\mathbb{Q}$ to functions on $\mathbb{R}$ (qns 46–48) 192
Summary 194

Historical note 195
Answers 197

8 Derivatives: tangents 203
Definition of derivative (qns 1–9) 203
Sums of functions (qns 10, 11) 205
The product rule (qns 13–16) 205
The quotient rule (qn 17) 206
The chain rule (qn 18) 206
Differentiability and continuity (qns 12, 19–25) 207

Derived functions (qns 26–34) 211
Second derivatives (qns 35–38) 212
Inverse functions (qns 39–45) 213
Derivatives at end points (qn 46) 215
Summary 215

Historical note 216
Answers 219

9 Differentiation and completeness: Mean Value Theorems, Taylor's Theorem 224
Rolle's Theorem (qns 1–11) 224
An intermediate value theorem for derivatives (qn 12) 226
The Mean Value Theorem (qns 13–24) 226
Cauchy's Mean Value Theorem (qn 25) 230
de l'Hôpital's rule (qns 26–30) 230
Summary – Rolle's Theorem and Mean Value Theorem 232

The Second and Third Mean Value Theorems (qns 31–34) 234
Taylor's Theorem or *n*th Mean Value Theorem (qns 35, 36) 235
Maclaurin's Theorem (qns 37–46) 236
Summary – Taylor's Theorem 239

Historical note 240
Answers 243

10 Integration: the Fundamental Theorem of Calculus 251
Areas with curved boundaries (qns 1–6) 251
Monotonic functions (qns 7–9) 254
The definite integral (qns 10–11) 255
Step functions (qns 12–15) 256
Lower integral and upper integral (qns 16–22) 258
The Riemann integral (qns 23–25) 260
Summary – definiton of the Riemann integral 261

Theorems on integrability (qns 26–36) 262
Integration and continuity (qns 37–44) 265
Mean Value Theorem for integrals (qn 45) 267
Integration on subintervals (qns 46–48) 267
Summary – properties of the Riemann integral 267

Indefinite integrals (qns 49–53) 268
The Fundamental Theorem of Calculus (qns 54–56) 270
Integration by parts (qns 57–59) 270
Integration by substitution (qn 60) 271

Improper integrals (qns 61–68) 271
Summary – the Fundamental Theorum of Calcalus 273

Historical note 273
Answers 276

11 Indices and circle functions 286
Exponential and logarithmic functions 286
Positive integers as indices (qns 1–3) 286
Positive rationals as indices (qns 4–7) 287
Rational numbers as indices (qns 8–17) 287
Real numbers as indices (qns 18–24) 289
Natural logarithms (qns 25–31) 290
Exponential and logarithmic limits (qns 32–38) 291
Summary – exponential and logarithmic functions 292

Circular or trigonometric functions 293
Length of a line segment (qns 39–42) 294
Arc length (qns 43–48) 295
Arc cosine (qns 49, 50) 296
Cosine and sine (qns 51–58) 297
Tangent (qns 59–62) 298
Summary – circular or trigonometric functions 298

Historical note 299
Answers 302

12 Sequences of functions 309
Pointwise limit functions (qns 1–14) 310
Uniform convergence (qns 15–19) 312
Uniform convergence and continuity (qns 20–23) 313
Uniform convergence and integration (qns 24–31) 315
Summary – uniform convergence, continuity and integration 317

Uniform convergence and differentiation (qns 32–34) 318
Uniform convergence of power series (qns 35–44) 319
The Binomial Theorem for any real index (qn 45) 322
The blancmange function (qn 46) 322
Summary – differentiation and the M-test 326

Historical note 326
Answers 328

Appendix 1 Properties of the real numbers 337

Appendix 2 Geometry and intuition 340

Appendix 3 Questions for student investigation and discussion 342

Bibliography 346
Index 351

1

Mathematical induction

Prelimary reading: Rosenbaum, Polya ch. 7.
Concurrent reading: Sominskii.
Further reading: Thurston chs 1 and 2, Ledermann and Weir.

The set of all whole numbers $\{1, 2, 3, \ldots\}$ is often denoted by $\mathbb{N}$. We will usually call $\mathbb{N}$ the set of *natural numbers*, and, sometimes, the set of *positive integers*. Note that $\mathbb{N}$ excludes 0.

1 $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Is this proposition true or false?
Test it when $n = 1$, $n = 2$ and $n = 3$.
How many values of n should you test if you want to be sure it is true for all natural numbers n?

If we write $f(n) = \dfrac{n(n+1)(2n+1)}{6}$, show that

$f(n) + (n+1)^2 = f(n+1)$.
Now suppose that the proposition with which we started holds for some particular value of n: add $(n+1)^2$ to both sides of the equation and deduce that the proposition holds for the next value of n.
Since we have already established that the proposition holds for $n = 1$, 2, and 3, the argument we have just formulated shows that it must hold for $n = 4$, and then by the same argument for $n = 5$, and so on.

2 When n is a natural number, $6^n - 5n + 4$ is divisible by 5.

Check this proposition for $n = 1$, 2 and 3.
By examining the difference between this number and $6^{n+1} - 5(n+1) + 4$, show that if the proposition holds for one value of n, it holds for the succeeding value of n. When you have done this you have established the two components of the proof of the proposition by mathematical induction.

3 Prove the following propositions by induction (some have easy alternative proofs which do not use induction):

(i) $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$,

(ii) $a + (a + d) + (a + 2d) + \ldots + (a + (n-1)d)$
$= \frac{1}{2}n(2a + (n-1)d)$,

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3 = [\frac{1}{2}n(n+1)]^2$,

(iv) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$,

(v) $+ 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$,

(vi) $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{x^n - 1}{x - 1}$, provided $x \neq 1$.

(vii) (optional) $1 + 2x + 3x^2 + \ldots + nx^{n-1} = \dfrac{nx^n}{x-1} - \dfrac{x^n - 1}{(x-1)^2}$,
provided $x \neq 1$.

4 Pascal's triangle, shown here, is defined inductively, each entry being the sum of the two (or one) entries in the preceding row nearest to the new entry. The $(r+1)$th entry in the nth row is denoted by $\binom{n}{r}$

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & 1
\end{array}$$

and called 'n choose r' because it happens to count the number of ways of choosing r objects from a set of n objects. Notice that we always have $0 \leq r \leq n$. From the portion of Pascal's triangle which has been shown, for example,

$$\binom{3}{0} = 1, \binom{3}{1} = 3, \binom{3}{2} = 3 \text{ and } \binom{3}{3} = 1.$$

From the definition of Pascal's triangle we have

$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, when $0 < r < n$.

Prove by induction that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$, taking $0! = 1$.

Verify that this formula gives $\binom{n}{r} = \binom{n}{n-r}$.

What aspect of Pascal's triangle does this reflect?

5 *The Binomial Theorem for positive integral index*
Prove by induction that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n.$$

6 Examine each of the propositions:

(i) $2^{n-1} \leq n^2$,
(ii) $n^2 + n + 41$ is a prime number.

Find values of n for which these propositions hold and also a value of n for which each is false.
If you were to attempt a proof by induction of either of these propositions, where would the proof break down?

7 Examine the numbers $6^n - 5n + 1$ and $6^{n+1} - 5(n+1) + 1$.
Find their difference.
Deduce that if the first of these numbers were divisible by 5 then the second would also be divisible by 5. Deduce also that if the second were divisible by 5 then the first would also be divisible by 5.
Is the first number divisible by 5 when $n = 1$?
Are there any values of n for which these numbers are divisible by 5?
If you were to attempt a proof by induction that the first number was divisible by 5, where would the proof break down?

8 Define $y^{(0)} = y$, $y^{(1)} = \dfrac{dy}{dx}$, and $y^{(n+1)} = \dfrac{dy^{(n)}}{dx}$.

(i) Let $y = \log_e(1+x)$, so $\dfrac{dy}{dx} = \dfrac{1}{1+x}$.

Prove that for $n \geq 1$, $(1+x)y^{(n+1)} + ny^{(n)} = 0$.
Deduce that when $x = 0$, $y^{(n+1)} = (-1)^n n!$

(ii) Let $y = \arctan x$, so $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.

Prove that for $n \geq 1$,
$(1+x^2)y^{(n+2)} + 2(n+1)xy^{(n+1)} + n(n+1)y^{(n)} = 0$.
Deduce that when $x = 0$, $y^{(2n)} = 0$ and $y^{(2n+1)} = (-1)^n(2n)!$

There is ambiguity in the literature on the question of whether $\mathbb{N}$ contains 0 or not. If mathematics starts with sets, and particularly the empty set, the equipment for counting the elements of finite sets must include 0. The psychological origins of counting, however, start with 1. It is this convention we have called *natural*. Both conventions are well-established.

THE PRINCIPLE OF MATHEMATICAL INDUCTION

A proposition, $P(n)$, relating to a natural number n, is valid for all natural numbers n, if

(a) $P(1)$ is true; the proposition is valid for $n = 1$ and

(b) $P(n) \Rightarrow P(n+1)$; the proposition for n implies the proposition for $n+1$.

Historical note

Triangular numbers were studied by Nicomachus (c. 100 A.D.) The sum of the first n squares was known to Archimedes (c. 250 B.C.) and the sum of the first n cubes to the Arabs (c. A.D. 1010). The justification of these forms uses induction implicitly.

The earliest use of proof by mathematical induction in the literature is by Maurolycus in his study of polygonal numbers published in Venice in 1575. Pascal knew the method of Marolycus and used it for work on the binomial coefficients (c. 1657). In 1713, Jacques Bernoulli used an inductive proof to make rigorous the claim of Wallis that

$$\sum_{r=0}^{r=n} \frac{r^k}{n^{k+1}} \to \frac{1}{k+1} \text{ as } n \to \infty.$$

By the latter part of the eighteenth century, induction was being used by several authors. The name 'mathematical induction' as distinct from scientific induction is due to De Morgan (1838). The use of inductive definitions long pre-dates this method of proof.

The well-ordering principle, that every non-empty set of natural numbers has a least member, is equivalent to mathematical induction (see Ledermann and Weir, 1996). Well-ordering was used by Euclid

(VII.31) to show that every integer has a prime factor and by Fermat (1637) in his proof of the non-existence of integral solutions to $x^4 + y^4 = z^2$. However, the recognition of the relationship between well-ordering and induction is recent, and from an historical point of view, awareness of the two principles developed independently.

During the nineteenth century there were several attempts to give a formal description of the natural numbers, notably by H. Grassmann (1861) and H. von Helmholtz (1887) both of whom assumed the Principle of Induction. R. Dedekind (1888) defined $\mathbb{N}$ as an arbitrary infinite chain and was able to prove the Principle of Induction. But in 1889, G. Peano, working with Dedekind's material, adopted the Principle of Mathematical Induction as one of his five postulates for defining the natural numbers, and his account remains standard to this day though Peano's Postulates admit non-standard models.

Answers and comments

1 Let $P(n)$ mean '$1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$'.
You checked that $P(1)$, $P(2)$ and $P(3)$ were all true.
The algebra you did showed that $P(n) \Rightarrow P(n+1)$.
So (by induction) $P(n)$ is true for all natural numbers n.

2 Let $P(n)$ mean '$6^n - 5n + 4$ is divisible by 5'.
Because $5 \mid 5$, $P(1)$ is true. Because $5 \mid 30$, $P(2)$ is true. Because $5 \mid 205$, $P(3)$ is true.
The difference between the expression for $n+1$ and the expression for n is $5 \cdot 6^n - 5$, which is divisible by 5, so $P(n) \Rightarrow P(n+1)$. And (by induction) $P(n)$ is true for all natural numbers n.

3 Take each of the six propositions in turn as $P(n)$. Verify the truth of $P(1)$ in each case. The algebra behind the proof of $P(n) \Rightarrow P(n+1)$ is as follows:

(i) $\frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n+1)(n+2)$,

(ii) $\frac{1}{2}n[2a + (n-1)d] + (a + nd) = \frac{1}{2}(n+1)(2a + nd)$,

(iii) $[\frac{1}{2}n(n+1)]^2 + (n+1)^3 = [\frac{1}{2}(n+1)(n+2)]^2$,

(iv) $\frac{1}{3}n(n+1)(n+2) + (n+1)(n+2) = \frac{1}{3}(n+1)(n+2)(n+3)$,

(v) $(2^n - 1) + 2^n = 2^{n+1} - 1$,

(vi) $$\frac{x^n - 1}{x - 1} + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

(vii) $$\frac{nx^n}{x-1} - \frac{x^n - 1}{(x-1)^2} + (n+1)x^n = \frac{nx^n + (n+1)x^n(x-1)}{x-1} - \frac{x^n - 1}{(x-1)^2}$$
$$= \frac{(n+1)x^{n+1}}{x-1} - \frac{x^{n+1} - 1}{(x-1)^2}.$$

4 If $P(n)$ is the proposition we are asked to prove for $0 \leq r \leq n$,

$$\binom{n}{0} = \binom{n}{n} = 1$$

is easily checked, and incorporates the truth of $P(1)$. If we assume $P(n)$ and apply it to the Pascal triangle property, we get $P(n+1)$.
Symmetry about a vertical axis.

5 For the inductive step, one must show that the coefficient of x^r in

$(1+x)^{n+1} = (1+x)^n(1+x)$ is $\binom{n}{r-1} + \binom{n}{r}$.

6 (i) True for $n \leq 6$, false for $7 \leq n$.

(ii) True for $n \leq 39$, false for $n = 40$, or a multiple of 41, or $40 + k^2$.

$P(n)$ does not imply $P(n + 1)$.

7 $6^n - 5n + 1$ divisible by $5 \Leftrightarrow 6^{n+1} - 5(n + 1) + 1$ divisible by 5.
But both statements are false because the first statement is false when $n = 1$. $P(1)$ is false.

8 (i) $(1 + x)dy/dx = 1$, or $(1 + x)y^{(1)} = 1$, so $(1 + x)y^{(2)} + y^{(1)} = 0$, and the proposition holds for $n = 1$.
Suppose $(1 + x)y^{(n+1)} + ny^{(n)} = 0$, differentiating we have $(1 + x)y^{(n+2)} + (n + 1)y^{(n+1)} = 0$, and the proposition for n implies the proposition for $n + 1$.
When $x = 0$, $y^{(1)} = 1$ and $y^{(n+1)} = -ny^{(n)}$. So $y^{(n+1)} = (-1)^n n! \Rightarrow y^{(n+2)} = (-1)^{n+1}(n + 1)!$ and the result holds by induction.

(ii) Differentiate $(1 + x^2)y^{(1)} = 1$ twice to get result for $n = 1$.
Differentiate result for n to obtain result for $n + 1$. Result follows by induction.
When $x = 0$, $y^{(1)} = 1$, $y^{(2)} = 0$ and $y^{(n+2)} + n(n + 1)y^{(n)} = 0$. So result holds for $n = 1$.
Also $y^{(2n)} = 0 \Rightarrow y^{(2n+2)} = 0$, and $y^{(2n+1)} = (-1)^n(2n)!$
$\Rightarrow y^{(2n+3)} = -(2n + 1)(2n + 2)(-1)^n(2n)! = (-1)^{n+1}(2n + 2)!$.
So result holds by induction.